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Complex Lagrangians and phantom cosmology

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Abstract

Motivated by the generalization of quantum theory for the case of non-Hermitian Hamiltonians with PT symmetry, we show how a classical cosmological model describes a smooth transition from ordinary dark energy to the phantom one. The model is based on a classical complex Lagrangian of a scalar field. Specific symmetry properties analogous to PT in non-Hermitian quantum mechanics lead to purely real equations of motion.

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1. Introduction

Complex (non-Hermitian) Hamiltonians with PT symmetry have been vigorously investigated in quantum mechanics and quantum field theory [1]. A possibility of applications to quantum cosmology has been pointed out in [2]. In the present contribution we focus attention on complex classical field theory. We explore the use of a particular complex scalar field Lagrangian, which has real solutions of the classical equations of motion. Thereby we provide a cosmological model describing in a natural way an evolution from the big bang to the big rip involving the transition from normal matter to phantom matter, crossing smoothly the phantom divide line.

The interest of our approach is related to its focusing on the intersection between two important fields of research. The basic idea can be introduced qualitatively as follows: given a complex Lagrangian of a complex scalar field, with the complex potential $V(\phi)$, if $V(\phi)$ is PT symmetric in the sense that $V(x)$ is PT symmetric in quantum mechanics, e.g. $V(\phi) \sim \exp(i\alpha\phi)$ with α real, then Lagrangian becomes real for purely imaginary ϕ and furthermore the kinetic term acquires a negative sign. In order to avoid any confusion we stress that our arguments are qualitative and PT symmetry is not to be intended in a literal way since we will deal with spatially homogeneous scalar fields and we have introduced these notations to refer the reader to the type of potentials and to their symmetry properties we will consider.

The framework in which we work is general relativity and classical cosmology. We will provide a self-contained introduction to these topics in order that an average reader can follow the presentation without too many difficulties.

2. Complex Lagrangians in classical field theory and cosmology

Let us consider a non-Hermitian (complex) Lagrangian of a scalar field

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi^* - V(\phi, \phi^*), \quad (1)$$

with the corresponding action,

$$S(\phi, \phi^*, g) = \int d^4x \sqrt{-\|g\|} \left(L + \frac{1}{6} R(g) \right), \quad (2)$$

where $\|g\|$ stands for the determinant of a metric $g^{\mu\nu}$ and $R(g)$ is the scalar curvature term and the Newton gravitational constant is normalized to $3/8\pi$ to simplify the Friedmann equations further on.

We employ potentials $V(\phi, \phi^*)$ satisfying the invariance condition

$$(V(\phi, \phi^*))^* = V(\phi^*, \phi), \quad (3)$$

while the condition

$$(V(\phi, \phi^*))^* = V(\phi, \phi^*) \quad (4)$$

is not satisfied. For example, such a potential can have a form

$$V(\phi, \phi^*) = V_1(\phi + \phi^*) V_2(\phi - \phi^*), \quad (5)$$

where V_1 and V_2 are real functions of their arguments. If one defines

$$\phi_1 \equiv \frac{1}{2}(\phi + \phi^*), \quad (6)$$

and

$$\phi_2 \equiv \frac{1}{2i}(\phi - \phi^*), \quad (7)$$

one can consider potentials of the form

$$V(\phi, \phi^*) = V_0(\phi_1) \exp(i\alpha\phi_2), \quad (8)$$

where α is a real parameter. In the last equation one can recognize the link to the so-called *PT* symmetric potentials.

Here, the functions ϕ_1 and ϕ_2 appear as the real and the imaginary parts of the complex scalar field ϕ ; however, in what follows, we shall treat them as independent spatially homogeneous complex variables depending only on the time parameter t .

We shall consider a flat spatially homogeneous Friedmann universe with the metric

$$ds^2 = dt^2 - a^2(t) dl^2, \quad (9)$$

satisfying the Friedmann equation

$$h^2 = \frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 + V_0(\phi_1) \exp(i\alpha\phi_2). \quad (10)$$

Here the variable $a(t)$ represents a cosmological radius of the universe and the Hubble variable $h(t) \equiv \frac{\dot{a}}{a}$ characterizes the velocity of expansion of the universe. The Friedmann equation (10) is nothing but Einstein equation, for the universe filled by scalar fields.

The equations of motion for fields ϕ_1 and ϕ_2 have the form

$$\ddot{\phi}_1 + 3h\dot{\phi}_1 + V_0'(\phi_1) \exp(i\alpha\phi_2) = 0, \quad (11)$$

$$i\ddot{\phi}_2 + 3ih\dot{\phi}_2 - \alpha V_0(\phi_1) \exp(i\alpha\phi_2) = 0. \quad (12)$$

Equations (10)–(12) are obtained by the variation of the action (2) with the Lagrangian (1) and the potential (8) with respect to the metric, and the scalar field variables ϕ_1 and ϕ_2 .

3. Cosmological solutions for accelerated universes

Let us note that the system of equations (10)–(12) can have a solution where $\phi_1(t)$ is real, while ϕ_2 is imaginary, or, in other words

$$\phi_2(t) = -i\xi(t), \quad (13)$$

where $\xi(t)$ is a real function. In terms of these two functions, our system of two real equations can be rewritten as

$$\ddot{\phi}_1 + 3\sqrt{\frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1) \exp(\alpha\xi)} \dot{\phi}_1 + V_0'(\phi_1) \exp(\alpha\xi) = 0, \quad (14)$$

$$\ddot{\xi} + 3\sqrt{\frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1) \exp(\alpha\xi)} \dot{\xi} - \alpha V_0(\phi_1) \exp(\alpha\xi) = 0. \quad (15)$$

Now, substituting $\phi_2(t)$ from equation (13) into equation (10) we have the following expression for the energy density:

$$\varepsilon = h^2 = \frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 + V_0(\phi_1) \exp(\alpha\xi). \quad (16)$$

Thus, we have paved the way to convert an action with a complex potential into the action with real potential and hyperbolic structure of the kinetic term. Note that in the flat Friedmann universe the energy density is always positive, being proportional to squared Hubble variable which should be real because of the reality of geometry.

Let us emphasize that the solution which we are looking for, namely, the imaginary solution for ϕ_2 is the solution which can have sense for the whole system of equations of Klein–Gordon type (11), (12) and that of Friedmann (10), because it makes the energy density real and positive. One can trace some kind of analogy with the non-Hermitian PT symmetric quantum theory with real spectra bounded from below. Our motivation is just to combine the idea of (i) providing physical application for ‘PT’-symmetric potentials and of (ii) building of a proper framework for conversion of the elliptic structure of the kinetic term for the scalar field to the hyperbolic one. In fact, it is crucial to start from a complex Lagrangian with the above-mentioned symmetry in order that equations (11) and (12) become real after the rotation $\phi_2(t) = -i\xi(t)$.

We shall see in section 4 that one can construct such solutions so that the originally complex Lagrangian becomes real on classical configurations while one of the scalar components obtains the ghost (negative) sign of kinetic energy. Thereby we recover a more conventional phantom matter starting from the complex matter with normal kinetic energy.

Now, coming back to the Einstein equations we should remember that the pressure will be equal

$$p = \frac{1}{2}\dot{\phi}_1^2 - \frac{1}{2}\dot{\xi}^2 - V_0(\phi_1) \exp(\alpha\xi). \quad (17)$$

The pressure in cosmology can be negative. Moreover, the so-called dark energy, responsible for the recently discovered phenomenon of cosmic acceleration [3], is characterized by the negative pressure such that $w \equiv p/\varepsilon < -1/3$ [4]. The value $w = -1$ is nothing but the cosmological constant, while the dark matter with $w < -1$ was dubbed as phantom energy [5]. The phantom models have some unusual properties: to realize them one often uses the phantom scalar field with the negative sign of kinetic term; in many models the presence of the phantom dark energy implies the existence of the future big rip cosmological singularity [6]. A cosmological evolution where dark energy undergoes the transition from $w > -1$ to $w < -1$ implies some particular properties of the corresponding field-theoretical model and is called crossing of phantom divide line [7].

From now on we show that the model introduced above is suitable for the description of the phenomenon of the phantom divide line crossing (for further detail, see [8]).

4. Crossing of the phantom divide line

It is easy to see that if $\dot{\phi}_1^2 < \xi^2$ the pressure will be negative and $p/\varepsilon < -1$, satisfying the phantom equation of state. Instead, when $\dot{\phi}_1^2 > \xi^2$, the ratio between the pressure and energy density exceeds -1 and, hence, the condition

$$\dot{\phi}_1^2 = \xi^2 \quad (18)$$

corresponds exactly to the phantom divide line, which can be crossed dynamically during the evolution of the field components $\phi_1(t)$ and $\xi(t)$.

We now provide a simple realization of this idea by an exactly solvable cosmological model by implementing the technique for the construction of potentials for a given cosmological evolution [9]. It is convenient to start with a cosmological evolution as given by the following expression for the Hubble variable:

$$h(t) = \frac{A}{t(t_R - t)}. \quad (19)$$

The evolution begins at $t = 0$, which represents a standard initial big bang cosmological singularity, and comes to an end in the big rip type singularity at $t = t_R$. The derivative of the Hubble variable

$$\dot{h} = \frac{A(2t - t_R)}{t^2(t_R - t)^2} \quad (20)$$

vanishes at

$$t_P = \frac{t_R}{2} \quad (21)$$

when the universe crosses the phantom divide line.

Next, we can write the standard formulae connecting the energy density and the pressure to the Hubble variable and its time derivative:

$$\frac{\dot{\phi}_1^2}{2} - \frac{\xi^2}{2} + V_0(\phi_1) e^{\alpha\xi} = h^2 = \frac{A^2}{t^2(t_R - t)^2}, \quad (22)$$

$$\frac{\dot{\phi}_1^2}{2} - \frac{\xi^2}{2} - V_0(\phi_1) e^{\alpha\xi} = -\frac{2}{3}\dot{h} - h^2 = -\frac{A(4t - 2t_R + 3A)}{3t^2(t_R - t)^2}. \quad (23)$$

The expression for the potential $V_0(\phi_1)$ follows

$$V_0(\phi_1) = \frac{A(2t - t_R + 3A)}{3t^2(t_R - t)^2} e^{-\alpha\xi}. \quad (24)$$

The kinetic term satisfies the equation

$$\dot{\phi}_1^2 - \xi^2 = -\frac{2A(2t - t_R)}{3t^2(t_R - t)^2}. \quad (25)$$

It is convenient to begin the construction with the solution for ξ . Taking into account formulae (19) and (24) equation (15) can be rewritten as

$$\ddot{\xi} + 3\dot{\xi} \frac{A}{t(t_R - t)} - \frac{\alpha A(2t - t_R + 3A)}{3t^2(t_R - t)^2} = 0. \quad (26)$$

Introducing a new parameter

$$m \equiv \frac{3A}{t_R}, \quad (27)$$

equation (26) looks like

$$\dot{y} + y \frac{mt_R}{t(t_R - t)} - \frac{\alpha mt_R(2t + t_R(m - 1))}{9t^2(t_R - t)^2} = 0, \quad (28)$$

where

$$y \equiv \dot{\xi}. \quad (29)$$

The solution of equation (28) is given by

$$y = \frac{\alpha m t_R (t_R - t)^m}{9 t^m} \int dt \frac{(2t + (m - 1)t_R)t^{m-2}}{(t_R - t)^{m+2}}. \quad (30)$$

Before considering the concrete values of m , note that the equation of state parameter w in the vicinity of the initial big bang singularity behaves as

$$w = -1 + \frac{2}{m}, \quad (31)$$

while approaching the final big rip singularity this parameter behaves as

$$w = -1 - \frac{2}{m}. \quad (32)$$

Note that the range for w does not depend on α , depending only on the value of the parameter m , which relates the scales of the Hubble variable h and of the time of the existence of the universe t_R .

Remarkably, the integral on the right-hand side of equation (30) is calculable analytically

$$\dot{\xi} = \frac{\alpha m t_R}{9 t (t_R - t)} \quad (33)$$

while

$$\xi = \frac{\alpha m}{9} (\log t - \log(t_R - t)). \quad (34)$$

From now on the parameter t will be dimensionless. The inclusion of characteristic time does not change the structure of the potential because of its exponential dependence on ξ . Substituting expression (33) into equation (25) one has

$$\dot{\phi}_1^2 = \frac{m t_R ((\alpha^2 m + 18)t_R - 36t)}{81 t^2 (t_R - t)^2}. \quad (35)$$

For the case $\alpha^2 m = 18$ the function $\phi_1(t)$ can be easily found from equation (35) and it looks like

$$\phi_1 = \pm \frac{4\sqrt{m}}{3} \operatorname{Arctanh} \sqrt{\frac{t_R - t}{t_R}}. \quad (36)$$

One can choose the positive sign in equation (36) without loss of generality.

Inverting equation (36) we obtain the dependence of the time parameter as a function of ϕ_1

$$t = \frac{t_R}{\cosh^2 \frac{3\phi_1}{4\sqrt{m}}}. \quad (37)$$

Substituting expressions (37) and (34) into equation (24) we can obtain the explicit expression for the potential $V_0(\phi_1)$:

$$V_0(\phi_1) = \frac{2 \cosh^6 \frac{3\phi_1}{4\sqrt{m}} (2 + 17 \cosh^2 \frac{3\phi_1}{4\sqrt{m}})}{t_R^2}. \quad (38)$$

We would like to emphasize that this potential is real and even. It is interesting that the time dependence of $\phi_1(t)$ could also be found for an arbitrary value of the parameter m , but for $\alpha^2 m > 18$ this dependence cannot be reversed analytically and, hence, one cannot obtain the explicit form of the potential $V_0(\phi_1)$.

5. Concluding remarks and perspectives

The phantom model building has involved many different ideas. Here we have presented a rather simple and natural cosmological toy model, linked to and inspired by such an intensively developing branch of quantum mechanics and quantum field theory as the study of non-Hermitian, but CPT (or PT) symmetric models. Note that there is an analogy between the manner in which the complexity of the original Lagrangian with the standard kinetic term is transformed into phantom-like Lagrangian, which is real but has a negative kinetic energy term and the equivalence between PT -symmetric quantum Hamiltonians and Hermitian Hamiltonian with variable effective mass (see e.g. [10]).

In this paper we have focused essentially on the classical field theory with a complex potential satisfying the invariance property of equation (3). Beyond the classical limit one could speculate on how quantum fluctuations $\delta\phi \equiv \eta_1(x) + i\eta_2(x)$ may preserve the consistency of this approach. Assume that the fields of fluctuations respect the initial and final conditions on a classical solution, $\eta_1(0) = \eta_2(0) = \eta_1(t_R) = \eta_2(t_R) = 0$. Then the second variation of the action (2) reads

$$S^{(2)}(\eta_1, \eta_2) = \frac{1}{2} \int d^4x a^3(t) \eta^T (-\partial_t^2 - 3h(t)\partial_t - \hat{V}^{(2)}(\phi_{1,c}, i\phi_{2,c}))\eta, \quad (39)$$

where $\eta^T = (\eta_1(x), \eta_2(x))$ is the transposed field, $\phi_{1,c} \equiv \phi_1(t)$, $\phi_{2,c} \equiv -i\xi(t)$ are classical solutions and the matrix $\hat{V}^{(2)}$ reads,

$$\hat{V}^{(2)} = \begin{pmatrix} \partial_{\phi_1}^2 V(\phi_1, \xi) & i\partial_{\phi_1} \partial_{\xi} V(\phi_1, \xi) \\ i\partial_{\phi_1} \partial_{\xi} V(\phi_1, \xi) & -\partial_{\xi}^2 V(\phi_1, \xi) \end{pmatrix}. \quad (40)$$

The quadratic form (39) is symmetric, with a non-Hermitian but pseudo-Hermitian matrix $\sigma_3 \hat{V}^{(2)} \sigma_3 = \hat{V}^{(2)\dagger}$ and the latter fact makes it possible to get real eigenvalues (or pairs of complex conjugated ones) of the energy operator in (39). Insofar as this energy operator is symmetric and has a 2×2 matrix form, one can diagonalize it⁴ with an (in general, non-local) orthogonal transformation \hat{O} so that $\hat{O}\hat{O}^T = \mathbf{I}$.

Because the second variation of potential $\hat{V}^{(2)}$ has complex matrix elements, the eigenvectors for a particular real eigenvalue will be also complex. The correct way to perform the variation is as follows:

- First, make an appropriate complex deformation of the integration contour of variables $\eta_1(x), \eta_2(x) \rightarrow \tilde{\eta}_1(x), \tilde{\eta}_2(x)$ so that the latter complex variables give rise to real ones $\eta = \hat{O}\tilde{\eta}$.
- Next, perform the diagonalization and end up with a well-defined, real energy operator, hopefully with positive masses.

To realize this program one has to solve the one-dimensional matrix Schrödinger-like equation. To give a concrete idea of what we have in mind we present a solvable example of potential (only mildly related to the previous discussion), $V = \exp(\sqrt{1 + \alpha^2}\phi_1) \exp(i\alpha\phi_2)$. Its second variation matrix is constant up to an overall factor,

$$\hat{V}^{(2)} = V \begin{pmatrix} 1 + \alpha^2 & i\alpha\sqrt{1 + \alpha^2} \\ i\alpha\sqrt{1 + \alpha^2} & -\alpha^2 \end{pmatrix}. \quad (41)$$

⁴ While it is not true in general for the complex symmetric matrix in arbitrary dimension as discussed for example in [11], in the case of a quadratic form in equation (39), one can straightforwardly show the above statement by explicit construction of the corresponding complex orthogonal 2×2 matrix.

Its eigenvalues are 0, 1 and the normalized eigenvectors are $e_0^T = (i\alpha, -\sqrt{1+\alpha^2})$, $e_1^T = (\sqrt{1+\alpha^2}, i\alpha)$ so that $e_i^T e_j = \delta_{ij}$. The diagonalization is realized by the complex orthogonal matrix

$$\hat{O} = \begin{pmatrix} i\alpha & -\sqrt{1+\alpha^2} \\ \sqrt{1+\alpha^2} & i\alpha \end{pmatrix}. \quad (42)$$

The above-mentioned contour is given by $\tilde{\eta} = \hat{O}^T \eta$ for arbitrary real η . Thus, in spite of non-Hermiticity of the matrix of second variation the relevant deformation of the contour makes the action real with positive kinetic terms and masses.

As a last remark we would like to point out that in quantum mechanics, models with CPT symmetry have been recently introduced in [12] with the difference that the charge operator is a differential operator contributing to a definition of a pseudometric operator, whereas in the present approach the ‘charge’ conjugation is related to a kind of internal degree of freedom.

Finally, let us recapitulate various steps of our approach.

1. We relate the possibility of a (de)-phantomization to a rotation of one of the components of the scalar field, which gives a hyperbolic structure to the kinetic term.
2. The preceding point provides compelling reasons to start from complex PT symmetric Lagrangian.
3. We succeed in reducing the problem to two coupled scalar fields: of course one could have started directly from these coupled equations, but in this case, underlying CPT symmetry and CP breaking would be hidden.

We do not claim uniqueness of our approach (see e.g. the recent paper [13], based on the superalgebraic approach and using Grassmann vector fields as well as a more general complexification involving the spacetime coordinates [14]), but a definite consequence turns out to be the relation between possible phantomization and CP-breaking.

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